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Multiple objective fractional programming involving semilocally type I-preinvex and related functions

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Received 13 February 2004

Available online 25 March 2005

Submitted by William F. Ames

Abstract

Sufficient optimality conditions are obtained for a nonlinear multiple objective fractional programming problem involving η-semidifferentiable type I-preinvex and related functions. Furthermore, a general dual is formulated and duality results are proved under the assumptions of generalized semilocally type I-preinvex and related functions. Our result generalize the results of Preda [V. Preda, Optimality and duality in fractional multiple objective programming involving semilocally preinvex and related functions, J. Math. Anal. Appl. 288 (2003) 365–382] and Stancu-Minasian [I.M. Stancu-Minasian, Optimality and duality in fractional programming involving semilocally preinvex and related functions, J. Inform. Optim. Sci. 23 (2002) 185–201].

1. Introduction

Optimality conditions and duality results for nonlinear multiple objective optimization have been the subject of much interest in the recent past and many contributions have

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doi:10.1016/j.jmaa.2005.02.032

been made to this development, e.g., Antczak [1], Ben-Israel and Mond [2], Cambini and Martin [3], Chankong and Haimes [4], Craven [5,6], Egudo [7], Elster and Nehse [8], Gupta and Vartak [10], Ivanov and Nehse [12], Jeyakumar [13], Jeyakumar and Mond [14], Kaul et al. [18], Mangasarian [19], Mishra [20–22], Mishra and Giorgi [23], Mishra et al. [24], Mishra and Mukherjee [25–27], Mishra and Rueda [28], Mishra et al. [29–31], Mititelu [32], Singh [40], Rueda and Hanson [42], Tanino and Sawaragi [44], Weir [33,45], Weir and Mond [46] and Yang et al. [47–49].

Jeyakumar [13] discussed a class of nonsmooth nonconvex problems in which functions are locally Lipschitz and are satisfying some invex type conditions. Mishra and Giorgi [23] extended this study to more general class of functions, namely semi-univex functions. For more details on nonsmooth programming problems the reader is referred to [15,21,23–26,30].

Elster and Nehse [8] considered a class of convex-like functions and obtained a saddle point optimality conditions for mathematical programs involving such functions. Ben-Israel and Mond [2] and Weir and Mond [46] considered a class of functions called preinvex functions. Jeyakumar and Mond [14] introduced a new class of generalized convex vector functions, called v-invex functions and some results on sufficiency and duality are obtained. Mishra and Mukherjee [21,26,30] extended this class of functions to the case of nonsmooth problems and obtained sufficiency and duality results for several problems. Furthermore, Mishra [22] and Mishra and Mukherjee [27] extended the class of v-invex functions to the case of continuous-time and established several duality results for variational and control problems.

Ewing [9] introduced semilocally convex functions which he applied it to derive sufficient optimality conditions for variational and control problems. Such functions have certain important convex type properties, e.g., local minima of semilocally convex functions defined on locally starshaped sets are also global minima, and nonnegative linear combinations of semilocally convex functions are also semilocally convex. Some generalizations of semilocally convex functions and their properties were investigated in Kaul and Kaur [16,17], Preda [36,37], Preda et al. [39], Stancu-Minasian [41], Suneja and Gupta [43], Mukherjee and Mishra [34,35]. Kaul and Kaur [17] derived sufficient optimality criteria for a class of nonlinear programming problems by using generalized semilocally functions. Optimality and duality results were given by Kaul and Kaur [17] for a nonlinear programming problem where the functions involved are semidifferentiable and generalized semilocally convex. Preda et al. [39] obtained optimality and duality results for nonlinear programming problems involving semilocally preinvex and related functions. Preda and Stancu-Minasian [38] extended the results of Preda et al. [39] to the multiple objective programming problems. Stancu-Minasian [41] established necessary and sufficient optimality conditions and duality results for nonlinear programming problems using semilocally preinvex and related functions. Preda [37] extended the results of Stancu-Minasian [41] to the multiple objective nonlinear problems.

It is well established in [18,42] that the class of type I and generalized type I functions are more general than that of the class of invex and generalized invex functions. Motivated by this and the work of Preda [36], we have extended the work of Preda [36] to the case of semilocally type I and related functions. Our results generalize the results obtained in the literature on this topic.

2. Definitions and preliminaries

For $x, y \in R^n$, by $x \leqq y$ we mean $x_i \leqq y_i$ for all i, $x \leqslant y$ means $x_i \leqq y_i$ for all i and $x_j < y_j$ for at least one j, $1 \leqq j \leqq n$. By $x < y$ we mean $x_i < y_i$ for all i and by $x \nleqslant y$ we mean the negation of $x \leqslant y$.

Let $X_0 \subseteq R^n$ be a set and $\eta : X_0 \times X_0 \to R^n$ be a vector application. We say that X_0 is invex at $\bar{x} \in X_0$ if $\bar{x} + \lambda\eta(x, \bar{x}) \in X_0$ for any $x \in X_0$ and $\lambda \in [0, 1]$. We say that the set X_0 is invex if X_0 is invex at any $x \in X_0$.

We remark that if $\eta(x, \bar{x}) = x - \bar{x}$ for any $x \in X_0$ then X_0 is invex at $\bar{x}$ iff X_0 is a convex set at $\bar{x}$.

Definition 1. We say that the set $X_0 \subseteq R^n$ is an η-locally starshaped set at $x, \bar{x} \in X_0$, if for any $x \in X_0$, there exists $0 < a_\eta(x, \bar{x}) \leqq 1$ such that $\bar{x} + \lambda\eta(x, \bar{x}) \in X_0$ for any $\lambda \in [0, a_\eta(x, \bar{x})]$.

Definition 2 [37]. Let $f : X_0 \to R^n$ be a function, where $X_0 \subseteq R^n$ is an η-locally starshaped set at $\bar{x} \in X_0$. We say that f is:

(a) semilocally preinvex (slpi) at $\bar{x}$ if, corresponding to $\bar{x}$ and each $x \in X_0$, there exists a positive number $d_\eta(x, \bar{x}) \leqq a_\eta(x, \bar{x})$ such that $f(\bar{x} + \lambda\eta(x, \bar{x})) \leqq \lambda f(x) + (1 - \lambda) f(\bar{x})$ for $0 < \lambda < d_\eta(x, \bar{x})$;
(b) semilocally quasi-preinvex (slqpi) at $\bar{x}$ if, corresponding to $\bar{x}$ and each $x \in X_0$, there exists a positive number $d_\eta(x, \bar{x}) \leqq a_\eta(x, \bar{x})$ such that $f(x) \leqq f(\bar{x})$ and $0 < \lambda < d_\eta(x, \bar{x})$ implies $f(\bar{x} + \lambda\eta(x, \bar{x})) \leqq f(\bar{x})$.

Definition 3. Let $f : X_0 \to R^n$ be a function, where $X_0 \subseteq R^n$ is an η-locally starshaped set at $\bar{x} \in X_0$. We say that f is η-semidifferentiable at $\bar{x}$ if $(df)^+(\bar{x}, \eta(x, \bar{x}))$ exists for each $\bar{x} \in X_0$, where

$$(df)^+\big(\bar{x}, \eta(x, \bar{x})\big) = \lim_{\lambda \to 0^+} \frac{1}{\lambda}\big[f\big(\bar{x} + \lambda\eta(x, \bar{x})\big) - f(\bar{x})\big]$$

(the right derivative at $\bar{x}$ along the direction $\eta(x, \bar{x})$).

If f is η-semidifferentiable at any $\bar{x} \in X_0$, then f is said to be η-semidifferentiable on X_0.

Remark. If $\eta(x, \bar{x}) = x - \bar{x}$, the η-semidifferentiability is the semidifferentiability notion. As is given in [36], if a function is directionally differentiable, then it is semidifferentiable but the converse is not true.

Lemma 1. *Let $f : X_0 \to R^n$ be an η-semidifferentiable function at $\bar{x} \in X_0$. If f is slqpi at $\bar{x}$ and $f(x) \leqq f(\bar{x})$ then $(df)^+(\bar{x}, \eta(x, \bar{x})) \leqq 0$.*

Definition 4 [37]. We say that f is semilocally pseudo-preinvex (slppi) at $\bar{x}$ if for any $\bar{x} \in X_0$, $(df)^+(\bar{x}, \eta(x, \bar{x})) \geqq 0 \Rightarrow f(x) \geqq f(\bar{x})$.

If f is slppi at any $\bar{x} \in X_0$, then f is said to be slppi on X_0.

Definition 5. Let X and Y be two subsets of X_0 and $\bar{y} \in Y$. We say that Y is η-locally starshaped at $\bar{y}$ with respect to X if for any $x \in X$ there exists $0 < a_\eta(x, \bar{y}) \leqq 1$ such that $\bar{y} + \lambda\eta(x, \bar{y}) \in Y$ for any $0 \leqq \lambda \leqq a_\eta(x, \bar{y})$.

Definition 6. Let Y be η-locally starshaped at $\bar{y}$ with respect to X and f be an η-semi-differentiable function at $\bar{y}$. We say that f is:

(a) slppi at $\bar{y} \in Y$ with respect to X, if for any $x \in X$, $(df)^+(\bar{y}, \eta(x, \bar{y})) \geqq 0 \Rightarrow f(x) \geqq f(\bar{y})$;
(b) strictly semilocally pseudo-preinvex (sslppi) at $\bar{y} \in Y$ with respect to X, if for any $x \in X, x \neq y (df)^+(\bar{y}, \eta(x, \bar{y})) \geqq 0 \Rightarrow f(x) > f(\bar{y})$.

We say that f is (slppi) sslppi on Y with respect to X, if f is (slppi) sslppi at any point of Y with respect to X.

Definition 7 (Elster and Nehse [8]). A function $f : X_0 \to R^k$ is a convex-like function if for any $x, y \in X_0$ and $0 \leqq \lambda \leqq 1$, there is $z \in X_0$ such that

$$f(z) \leqq \lambda f(x) + (1 - \lambda) f(y).$$

Remark. The convex and the preinvex functions are convex-like functions.

Lemma 2 (Hayashi and Komiya [11]). *Let S be a nonempty set in R^n and $\psi : S \to R^k$ be a convex-like function. Then either*

$$\psi(x) < 0 \quad \textit{has a solution } x \in S$$

or

$$\lambda^T \psi(x) \geqq 0 \quad \textit{for all } x \in S,$$

for some $\lambda \in R^k$, $\lambda \geqslant 0$, but both alternatives are never true. (*Here the symbol T denotes the transpose of a matrix.*)

Using Lemma 2 from above instead of Lemma 2.9 from [38], we have that Theorems 3.4 and 3.5 stated there are still true. Thus, in the next section we will use the following version of Theorem 3.5 from [38].

Lemma 3. *Let $\bar{x} \in X$ be a* (*local*) *weak minimum solution for the following problem*:

$$\min\big(\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)\big)$$
$$\textit{subject to} \quad \begin{cases} h_j(x) \leqq 0, & j \in M, \\ x \in X_0, \end{cases}$$

where $\varphi = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_p(x)) : X_0 \to R^p$ and $h_1, \ldots, h_m$ are η-semidifferentiable at $\bar{x}$. Also, assume that h_j $(j \in N(\bar{x}))$ is a continuous function at $\bar{x}$ and $(d\varphi)^+(\bar{x}, \eta(x, \bar{x}))$

and $(dh)^+(\bar{x}, \eta(x, \bar{x}))$ *are convex-like functions of* $\bar{x}$ *on* X_0*. If* h *satisfies a regularity condition at* $\bar{x}$ *(see* [38]*), then there exist* $\lambda^0 \in R^p$*,* $u^0 \in R^m$ *such that*

$$\lambda^{0^T} (d\varphi)^+\big(\bar{x}, \eta(x, \bar{x})\big) + u^{0^T} (dh)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \quad \textit{for all } x \in X_0,$$
$$u^{0^T} h(\bar{x}) = 0, \quad h(\bar{x}) \leqq 0,$$
$$\lambda^{0^T} e = 1, \quad \lambda^0 \geqslant 0, \quad u^0 \geqq 0,$$

where $e = (1, 1, \ldots, 1)^T \in R^p$*.*

In this paper we consider the following multiple objective nonlinear fractional programming problem:

$$\text{(VFP)} \quad \min\left(\frac{f_1(x)}{g_1(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right)$$
$$\text{subject to} \quad \begin{cases} h_j(x) \leqq 0, & j = 1, 2, \ldots, m, \\ x \in X_0, \end{cases}$$

where $X_0 \subseteq R^n$ is a nonempty set and $g_i(x) > 0$ for all $x \in X_0$ and each $i = 1, \ldots, p$. Let $f = (f_1, \ldots, f_p)$, $g = (g_1, \ldots, g_p)$ and $h = (h_1, \ldots, h_m)$.

We put $X = \{x \in X_0\colon\ h_j(x) \leqq 0,\ \ j = 1, 2, \ldots, m\}$ for the feasible set of problem (VFP).

Definition 8. We say that the problem (VFP) is η-semidifferentiable type I-preinvex at $\bar{x}$ if for any $\bar{x} \in X_0$, we have

$$f_i(x) - f_i(\bar{x}) \geqq (df_i)^+\big(\bar{x}, \eta(x, \bar{x})\big), \quad \forall i \in P,$$
$$g_i(x) - g_i(\bar{x}) \leqq (dg_i)^+\big(\bar{x}, \eta(x, \bar{x})\big), \quad \forall i \in P,$$
$$-h_j(\bar{x}) \geqq (dh_j)^+\big(\bar{x}, \eta(x, \bar{x})\big), \quad \forall j \in M.$$

Definition 9. We say that the problem (VFP) is η-semidifferentiable pseudo-quasi-type I-preinvex at $\bar{x}$ if for any $x \in X_0$, we have

$$(df_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \quad \Rightarrow \quad f_i(x) \geqq f_i(\bar{x}), \quad \forall i \in P,$$
$$(dg_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \quad \Rightarrow \quad g_i(x) \leqq g_i(\bar{x}), \quad \forall i \in P,$$
$$-h_j(\bar{x}) \leqq 0 \quad \Rightarrow \quad (dh_j)^+\big(\bar{x}, \eta(x, \bar{x})\big) \leqq 0, \quad \forall j \in M.$$

The problem (VFP) is η-semidifferentiable pseudo-quasi-type I-preinvex on X_0 if it is η-semidifferentiable pseudo-quasi-type I-preinvex at any $\bar{x} \in X_0$.

Definition 10. We say that the problem (VFP) is η-semidifferentiable quasi-pseudo-type I-preinvex at $\bar{x}$ if for any $x \in X_0$, we have

$$f_i(x) \leqq f_i(\bar{x}) \quad \Rightarrow \quad (df_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) \leqq 0, \quad \forall i \in P,$$
$$g_i(x) \leqq g_i(\bar{x}) \quad \Rightarrow \quad (dg_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0, \quad \forall i \in P,$$
$$(dh_j)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \quad \Rightarrow \quad -h_j(\bar{x}) \geqq 0, \quad \forall j \in M.$$

The problem (VFP) is η-semidifferentiable quasi-pseudo-type I-preinvex on X_0 if it is η-semidifferentiable pseudo-quasi-type I-preinvex at any $\bar{x} \in X_0$.

Definition 11. For the problem (VFP), a point $\bar{x} \in X$ is said to be a weak minimum if there exists no other feasible point x for which $f(\bar{x})/g(\bar{x}) > f(x)/g(x)$.

For $\bar{x} \in X$ we put $M(\bar{x}) = \{j \in M\colon h_j(\bar{x}) = 0\}$, $h^0 = (h_j)_{j \in M(\bar{x})}$ and $N(\bar{x}) = M \backslash M(\bar{x})$, where $M = \{1, 2, \ldots, m\}$.

Definition 12. We say that (VFP) satisfies the generalized Slater's constraint qualification (GSCQ) at $\bar{x} \in X$ if h^0 is slppi at $\bar{x}$ and there exists an $\hat{x} \in X$ such that $h^0(\hat{x}) < 0$.

Lemma 4. *Let $\bar{x} \in X$ be a* (*local*) *weak minimum solution for* (VFP)*. Further, we assume that h_j is continuous at $\bar{x}$ for any $j \in N(\bar{x})$ and that f, g, h^0 are η-semidifferentiable at $\bar{x}$. Then, the system*

$$\begin{cases} (df)^+(\bar{x}, \eta(x, \bar{x})) < 0, \\ (dg)^+(\bar{x}, \eta(x, \bar{x})) > 0, \\ (dh^0)^+(\bar{x}, \eta(x, \bar{x})) < 0 \end{cases}$$

has no solution $x \in X_0$.

Lemma 5 (Fritz John type necessary optimality criteria). *Let us suppose that h_j $(j \in N(\bar{x}))$ is a continuous function at $\bar{x}$ and $(df)^+(\bar{x}, \eta(x, \bar{x}))$, $(dg)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dh^0)^+(\bar{x}, \eta(x, \bar{x}))$ are convex-like functions of x on X_0. If $\bar{x}$ is a* (*local*) *weak minimum solution for* (VFP)*, then there exist $\lambda^0 \in R^p$, $u^0 \in R^p$, $v^0 \in R^m$ such that*

$$\begin{aligned} &\lambda^{0^T} (df)^+\big(\bar{x}, \eta(x, \bar{x})\big) - u^{0^T} (dg)^+\big(\bar{x}, \eta(x, \bar{x})\big) + v^{0^T} (dh^0)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \\ &\quad \textit{for all } x \in X_0, \\ &v^{0^T} h(\bar{x}) = 0, \\ &(\lambda^0, u^0, v^0) \neq 0, \qquad (\lambda^0, u^0, v^0) \geqq 0. \end{aligned}$$

For each $u = (u_1, \ldots, u_p) \in R^p_+$, where R^p_+ denotes the positive orthant of R^p, we consider

$$(\mathrm{VFP}_u) \quad \min\big(f_1(x) - u_1 g_1(x), \ldots, f_p(x) - u_p g_p(x)\big)$$
$$\text{subject to} \quad \begin{cases} h_j(x) \leqq 0, & j \in M, \\ x \in X_0. \end{cases}$$

The following lemma is easy to prove.

Lemma 6. *If $\bar{x}$ is a* (*local*) *weak minimum for* (VFP) *then $\bar{x}$ is a* (*local*) *weak minimum for* (VFP^0_u)*, where $u^0 = f(\bar{x})/g(\bar{x})$.*

Using this lemma we can get the following Karush–Kuhn–Tucker type necessary optimality criterion for the problem (VFP).

Lemma 7 (Karush–Kuhn–Tucker type necessary optimality criterion). *Let $\bar{x}$ be a* (*local*) *weak minimum solution for* (VFP), *let h_j be a continuous at $\bar{x}$ for $j \in N(\bar{x})$ and let $(df_i)^+(\bar{x}, \eta(x, \bar{x}))$, $(dg_i)^+(\bar{x}, \eta(x, \bar{x}))$, $i \in P$, and $(dh^0)^+(\bar{x}, \eta(x, \bar{x}))$ be convex-like functions of x on X_0. If g satisfies* (GSCQ) *at $\bar{x}$, then there exist $\lambda^0 \in R^p_+$, $u^0 \in R^p_+$, $v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \big((df)^+\big(\bar{x}, \eta(x, \bar{x})\big) - u^{0^T}(dg)^+\big(\bar{x}, \eta(x, \bar{x})\big) + v^{0^T}(dh^0)^+\big(\bar{x}, \eta(x, \bar{x})\big)\big) \geqq 0$$

for all $x \in X_0$,

$$v^{0^T} h(\bar{x}) = 0,$$

$$h(\bar{x}) \leqq 0,$$

$$\lambda^{0^T} e = 1,$$

$$\lambda^0 \geqslant 0, \quad u^0 \geqq 0, \quad v^0 \geqq 0,$$

where $e = (1, \ldots, 1)^T \in R^p$.

Remark. In the lemma above we can suppose, for any $i \in P$, that $(df_i)^+(\bar{x}, \eta(x, \bar{x})) - u_i^0(dg_i)^+(\bar{x}, \eta(x, \bar{x}))$ is convex-like on X_0, where $u_i^0 = f_i(\bar{x})/g_i(\bar{x})$, instead of considering that $(df_i)^+(\bar{x}, \eta(x, \bar{x}))$ and $(dg_i)^+(\bar{x}, \eta(x, \bar{x}))$, $i \in P$, are convex-like on X_0, for any $i \in P$.

3. Sufficient optimality criteria

In this section, using the concept of (local) weak optimality, we give some sufficient optimality conditions for the (VFP) problem.

Theorem 1. *Let $\bar{x} \in X$ and* (VFP) *is η-semilocally type I-preinvex at $\bar{x}$. Further, we assume that there exists $\lambda^0 \in R^p$, $u^0 \in R^p$ and $v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0 \big((df_i)^+\big(\bar{x}, \eta(x, \bar{x})\big)\big) + v^{0^T}(dh)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0 \quad \text{for all } x \in X, \tag{3.1}$$

$$(dg_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) \leqq 0, \quad \forall x \in X,\ \forall i \in P, \tag{3.2}$$

$$v^{0^T} h(\bar{x}) = 0, \tag{3.3}$$

$$h(\bar{x}) \leqq 0, \tag{3.4}$$

$$\lambda^{0^T} e = 1, \tag{3.5}$$

$$\lambda^0 \geqslant 0, \quad u^0 \geqq 0, \quad v^0 \geqq 0, \tag{3.6}$$

where $e = (1, \ldots, 1)^T \in R^p$. Then $\bar{x}$ is a weak minimum solution for (VFP).

Proof. Suppose that the result does not hold. Hence there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \quad \text{for any } i \in P. \tag{3.7}$$

Since (VFP) is η-semilocally type I-preinvex at $\bar{x}$, we get

$$f_i(\tilde{x}) - f_i(\bar{x}) \geqq (df)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad i \in P, \tag{3.8}$$

$$g_i(\tilde{x}) - g_i(\bar{x}) \leqq (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad i \in P, \tag{3.9}$$

$$-h_j(\bar{x}) \geqq (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad j \in M. \tag{3.10}$$

Multiplying (3.8) by $\lambda_i^0 \geqq 0$, $i \in P$, $\lambda^0 \in R_+^p$, (3.10) by $v_j^0 \geqq 0$, $j \in M$, then summing the obtained relations and using (3.1), we get

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - f_i(\bar{x})\big) - \sum_{j=1}^{m} v_j^0 h_j(\bar{x}) \\
&\quad \geqq \sum_{i=1}^{p} \lambda_i^0 (df_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) + \sum_{J=1}^{m} v_j^0 (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) \geqq 0.
\end{aligned}$$

Hence,

$$\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - f_i(\bar{x})\big) - \sum_{j=1}^{m} v_j^0 h_j(\bar{x}) \geqq 0. \tag{3.11}$$

Since $x \in X$, $v^0 \geqq 0$, by (3.3) and (3.11), we get

$$\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - f_i(\bar{x})\big) \geqq 0. \tag{3.12}$$

Using (3.5), (3.6) and (3.12), we obtain that there exists $i_0 \in P$ such that

$$f_{i_0}(\tilde{x}) \geqq f_{i_0}(\bar{x}). \tag{3.13}$$

By (3.2) and (3.9) it follows that

$$g_i(\tilde{x}) \leqq g_i(\bar{x}), \quad i \in P. \tag{3.14}$$

Now using (3.13), (3.14) and $f \geqq 0$, $g > 0$, we obtain

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geqq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which is a contradiction to (3.7). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP). □

Theorem 2. *Let $\bar{x} \in X$ and* (VFP) *is η-semilocally type I-preinvex at $\bar{x}$. Further, we assume that there exists $\lambda^0 \in R^p$, $u_i^0 = f_i(\bar{x})/g_i(\bar{x})$, $i \in P$, and $v^0 \in R^m$ such that*

$$\sum_{i=1}^{p} \lambda_i^0\Big((df_i)^+\big(\bar{x}, \eta(x, \bar{x})\big) - u_i^0 (dg_i)^+\big(\bar{x}, \eta(x, \bar{x})\big)\Big) + v^{0^T} (dh)^+\big(\bar{x}, \eta(x, \bar{x})\big) \geqq 0,$$
$$\forall x \in X, \tag{3.15}$$

$$v^{0^T} h(\bar{x}) = 0, \tag{3.16}$$

$$h(\bar{x}) \leqq 0, \tag{3.17}$$

$$\lambda^{0^T} e = 1, \tag{3.18}$$

$$\lambda^0 \geqslant 0, \quad u^0 \geqq 0, \quad v^0 \geqq 0, \tag{3.19}$$

where $e = (1, \ldots, 1)^T \in R^p$. *Then* $\bar{x}$ *is a weak minimum solution for* (VFP).

Proof. Suppose that the result does not hold. Then if $\bar{x}$ is not a weak minimum solution for (VFP), we have that there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \quad \text{for any } i \in P,$$

that is,

$$f_i(\tilde{x}) < u_i^0 g_i(\tilde{x}) \quad \text{for any } i \in P. \tag{3.20}$$

Since (VFP) is η-semilocally type I-preinvex at $\bar{x}$, we get

$$f_i(\tilde{x}) - f_i(\bar{x}) \geqq (df)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad i \in P,$$
$$g_i(\tilde{x}) - g_i(\bar{x}) \leqq (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad i \in P,$$
$$-h_j(\bar{x}) \geqq (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big), \quad j \in M.$$

Using these inequalities (3.19) and (3.15), we get

$$\begin{aligned}
&\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - f_i(\bar{x})\big) - \sum_{i=1}^{p} \lambda_i^0 u_i^0\big(g_i(\tilde{x}) - g_i(\bar{x})\big) - \sum_{i=1}^{m} v_j^0 h_j(\bar{x}) \\
&\quad \geqq \sum_{i=1}^{p} \lambda_i^0\Big((df_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) - u_i^0 (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big)\Big) + \sum_{i=1}^{m} v_j^0 (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) \\
&\quad \geqq 0.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})\big) - \big(f_i(\bar{x}) - u_i^0 g_i(\bar{x})\big) - \sum_{i=1}^{m} v_j^0 h_j(\bar{x}) \geqq 0.$$

Since $u_i^0 = f_i(\bar{x})/g_i(\bar{x})$, $i \in P$, we obtain

$$\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})\big) - \sum_{i=1}^{m} v_j^0 h_j(\bar{x}) \geqq 0.$$

Since $\tilde{x} \in X$, $v^0 \geqq 0$, by (3.16) and (3.19), we get

$$\sum_{i=1}^{p} \lambda_i^0\big(f_i(\tilde{x}) - u_i^0 g_i(\tilde{x})\big) \geqq 0. \tag{3.21}$$

Since $\lambda_i^0 \geqq 0$, $\lambda^{0^T} e = 1$, we obtain that there exists $i_0 \in P$ such that

$$f_{i_0}(\tilde{x}) - u_{i_0}^0 g_{i_0}(\tilde{x}) \geqq 0,$$

that is,

$$\frac{f_{i_0}(\tilde{x})}{g_{i_0}(\tilde{x})} \geqq \frac{f_{i_0}(\bar{x})}{g_{i_0}(\bar{x})},$$

which is a contradiction to (3.15). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP). □

Theorem 3. *Let $\bar{x} \in X$, $\lambda^0 \in R^p$, $u_i^0 = f_i(\bar{x})/g_i(\bar{x})$, $i \in P$, and $v^0 \in R^m$ be such that the conditions* (3.15)–(3.19) *of Theorem* 2 *hold. Furthermore, we assume that* (VFP$_u$) *is η-semilocally pseudo-quasi-type I-preinvex at $\bar{x}$. Then $\bar{x}$ is a weak minimum solution for* (VFP$_u$).

Proof. Suppose that $\bar{x}$ is not a weak minimum solution for (VFP$_u$). Then there exists $\tilde{x} \in X$ such that

$$\frac{f_i(\tilde{x})}{g_i(\tilde{x})} < \frac{f_i(\bar{x})}{g_i(\bar{x})} \quad \text{for any } i \in P,$$

that is,

$$f_i(\tilde{x}) < u_i^0 g_i(\tilde{x}) \quad \text{for any } i \in P,$$

which is equivalent to

$$f_i(\tilde{x}) - u_i^0 g_i(\tilde{x}) < f_i(\bar{x}) - u_i^0 g_i(\bar{x}) \quad \text{for any } i \in P.$$

By the η-semilocally pseudo-type I-preinvexity at $\bar{x}$, of (VFP$_u$), we get

$$(df_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) - u_i^0 (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) < 0 \quad \text{for any } i \in P.$$

Using $\lambda_i^0 \in R_+^p$, $\lambda^{0^T} e = 1$, we obtain

$$\sum_{i=1}^{p} \lambda_i^0 \big((df_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) - u_i^0 (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big)\big) < 0. \tag{3.22}$$

By the η-semilocally quasi-type I-preinvexity at $\bar{x}$, of (VFP$_u$) and (3.16) and $v^0 \in R_+^m$ we get

$$\sum_{j=1}^{m} v_j^0 (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) \leqq 0. \tag{3.23}$$

Now, by (3.22) and (3.23) we obtain

$$\sum_{i=1}^{p} \lambda_i^0 \big((df_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) - u_i^0 (dg_i)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big)\big) + \sum_{j=1}^{m} v_j^0 (dh_j)^+\big(\bar{x}, \eta(\tilde{x}, \bar{x})\big) < 0,$$

which is a contradiction to (3.15). Thus, the theorem is proved and $\bar{x}$ is a weak minimum solution for (VFP$_u$). □

4. Duality

We consider, for (VFP), a general Mond–Weir dual (MWD) as

$$\max \psi(y, \lambda, u, v) = u - v_{I_0}^T h_{I_0}(y)e$$

subject to

$$\sum_{i=1}^{p} \lambda_i \big((df_i)^+\big(y, \eta(x, y)\big) - u_i (dg_i)^+\big(y, \eta(x, y)\big)\big) + v^T (dh)^+\big(y, \eta(x, y)\big) \geqq 0$$
$$\text{for all } x \in X, \tag{4.1}$$

$$f_i(y) - u_i g_i(y) \geqq 0 \quad \text{for any } i \in P, \tag{4.2}$$

$$v_{I_s}^T h_{I_s}(y) \geqq 0 \quad (1 \leqq s \leqq \gamma), \tag{4.3}$$

$$\lambda^T e = 1, \quad \lambda \geqslant 0, \quad \lambda \in R^p, \tag{4.4}$$

$$u \geqq 0, \quad u \in R^p, \quad v \geqq 0, y \in X_0, \tag{4.5}$$

where $\gamma \geqq 1$, $I_s \cap I_s = \emptyset$ for $s \neq t$ and $\bigcup_{s=0}^{\gamma} I_s = M$. (Here $v_{I_s} = (v_j)_{j \in I_s}$, $h_{I_s} = (h_j)_{j \in I_s}$.)

Let W denote the set of all feasible solutions of (FMWD). Also, we define the following sets:

$$A = \big\{(\lambda, u, v) \in R^p \times R^p \times R^m\colon\ (y, \lambda, u, v) \in W \text{ for some } y \in X_0\big\}$$

and, for $(\lambda, u, v) \in A$,

$$B(\lambda, u, v) = \big\{y \in X_0\colon\ (y, \lambda, u, v) \in W\big\}.$$

We put $B = \bigcup_{(\lambda,u,v)\in A} B(\lambda, u, v)$ and note that $B \subset X_0$. Also, we note that if $(y, \lambda, u, v) \in W$ then $(\lambda, u, v) \in A$ and $y \in B(\lambda, u, v)$.

Now we establish certain duality results between (VFP) and (FMWD). Assume that f, g and h are η-semidifferentiable on X.

Theorem 4 (Weak duality). *Assume that for all feasible solutions $x \in X$ and $(y, \lambda, u, v) \in W$ for* (VFP) *and* (FMWD), *respectively, and*

$$(df_i)^+\big(y, \eta(x, y)\big) - u_i (dg_i)^+\big(y, \eta(x, y)\big) + \sum_{j \in I_0} v_j (dh_j)^+\big(y, \eta(x, y)\big) \geqq 0$$
$$\Rightarrow \quad f_i(x) - u_i g_i(x) + v_{I_0}^T h_{I_0}(x) \geqslant f_i(y) - u_i g_i(y) + v_{I_0}^T h_{I_0}(y)$$
$$\textit{for all } i \in P \tag{4.6}$$

and

$$-v_{I_s}^T h_{I_s}(y) \leqq 0 \quad \Rightarrow \quad \sum_{j \in I_s} v_j (dh_j)^+\big(y, \eta(x, y)\big) \leqq 0, \quad 1 \leqq s \leqq \gamma, \tag{4.7}$$

hold on $B(\lambda, u, v)$. Then the following cannot hold:

$$f_i(x) - u_i g_i(x) \leqq v_{I_0}^T h_{I_0}(y) \quad \textit{for any } i \in P \tag{4.8}$$

and

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{I_0}^T h_{I_0}(y) \quad \textit{for some } i_0 \in P. \tag{4.9}$$

Proof. Using (4.3) and (4.7), we get

$$\sum_{j \in I_s} v_j (dh_j)^+\big(y, \eta(x, y)\big) \leqq 0, \quad 1 \leqq s \leqq \gamma. \tag{4.10}$$

Now we suppose contrary to the result of the theorem that (4.8) and (4.9) hold. Hence if (4.8) and (4.9) hold for some feasible $x \in X$ and $(y, \lambda, u, v) \in W$ for (VFP) and (FMWD), we obtain

$$f_i(x) - u_i g_i(x) \leqq v_{I_0}^T h_{I_0}(y) \quad \text{for any } i \in P \tag{4.11}$$

and

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{I_0}^T h_{I_0}(y) \quad \text{for some } i_0 \in P. \tag{4.12}$$

Using (4.2) and (4.5) and the feasibility of x for (VFP), we have

$$v_{I_0}^T h_{I_0}(x) \leqq 0 \leqq f_i(y) - u_i g_i(y) \quad \text{for any } i \in P. \tag{4.13}$$

From (4.11)–(4.13), we get

$$f_i(x) - u_i g_i(x) + v_{I_0}^T h_{I_0}(x) \leqq f_i(y) - u_i g_i(y) + v_{I_0}^T h_{I_0}(y) \quad \text{for any } i \in P \tag{4.14}$$

and

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) + v_{I_0}^T h_{I_0}(x) \leqq f_{i_0}(y) - u_{i_0} g_{i_0}(y) + v_{I_0}^T h_{I_0}(y) \quad \text{for some } i_0 \in P. \tag{4.15}$$

By (4.6), (4.14) and (4.15), we obtain

$$(df_i)^+\big(y, \eta(x, y)\big) - u_i (dg_i)^+\big(y, \eta(x, y)\big) + \sum_{j \in I_0} v_j (dh_j)^+\big(y, \eta(x, y)\big) < 0. \tag{4.16}$$

Now from (4.16) and (4.1), we get

$$\sum_{s=1}^{\gamma} \sum_{j \in I_s} v_j (dh_j)^+\big(y, \eta(x, y)\big) > 0,$$

which is a contradiction to (4.10). Thus the theorem is proved. □

Theorem 5 (Weak duality). *Assume that for all feasible solutions $x \in X$ and $(y, \lambda, u, v) \in W$ for* (VFP) *and* (FMWD), *respectively, and*

$$(df_i)^+\big(y, \eta(x, y)\big) - u_i (dg_i)^+\big(y, \eta(x, y)\big) + \sum_{j \in I_0} v_j (dh_j)^+\big(y, \eta(x, y)\big) \geqq 0$$
$$\Rightarrow \quad f_i(x) - u_i g_i(x) + v_{I_0}^T h_{I_0}(x) \geqq f_i(y) - u_i g_i(y) + v_{I_0}^T h_{I_0}(y) \quad \textit{for all } i \in P$$

and

$$-v_{I_s}^T h_{I_s}(y) \leqq 0 \quad \Rightarrow \quad \sum_{j\in I_s} v_j (dh_j)^+\big(y, \eta(x, y)\big) \leqq 0, \quad 1 \leqq s \leqq \gamma,$$

hold on $B(\lambda, u, v)$ *with* $\lambda > 0$. *Then the following cannot hold*:

$$f_i(x) - u_i g_i(x) \leqq v_{I_0}^T h_{I_0}(y) \quad \textit{for any } i \in P$$

and

$$f_{i_0}(x) - u_{i_0} g_{i_0}(x) < v_{I_0}^T h_{I_0}(y) \quad \textit{for some } i_0 \in P.$$

Proof. The proof is very similar to the proof of Theorem 4. □

Acknowledgment

Authors are thankful to the referee for his careful reading of the paper and constructive suggestions that helped to improve the final shape of the paper.

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